The Effect of a Reflective Interface along the Input Waveguide on the Q Measurement of a Resonate Cavity

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Abstract: *A method to calculate the error in the measured quality factor due to the presence of a reflective interface along the input waveguide coupled to a resonant cavity is presented and applied to the CKM (Charged Kaons at the Main Injector) Superconducting Radio Frequency (SCRF) cavity quality factor measurement system.*

Introduction

A reflective interface existing along the input waveguide coupled to a resonant cavity will interact with the reflection at the cavity/coupler interface. It's influence on the reflection coefficient and the cavity decay time will introduce an error into the calculation of the cavity's unloaded quality factor (Q). The specific case considered here is a reflective interface in the form of an adapter connecting the input transmission line to the input coupler of the CKM (Charged Kaons at the Main Injector) superconducting radio frequency (SCRF) deflecting mode cavity. A method to determine the error in the calculation of the cavity Q due to this adapter is presented and applied to the vertical test stand Q measurement system which has been used to characterize the CKM cavities.

The Measurement System Model

A circuit model of the measurement system is shown in Fig. 1. A length of lossy transmission line separates an adapter from a cavity that is coupled onto an input transmission line through a coupling coefficient of β[1]. For simplicity the transformer representing the coupling has not been shown, rather the cavity has already been transformed onto the input transmission line. The adapter is assumed to be a lossless reciprocal 2-port device[2] whose s-parameters vary slowly with frequency and whose reference planes are considered such that s11 is real and equals s22 for simplicity. Not shown is a pickup probe that is weakly coupled to the cavity. It monitors the energy in the cavity and is used to measure the decay time of the cavity/coupler system. The measurement reference plane is at the leftmost reference plane of the adapter. It is here that a source and directional coupler are inserted for measurements. The 50Ω resistor represents the impedance that the source presents to the system when the source is shut off for decay time measurement.

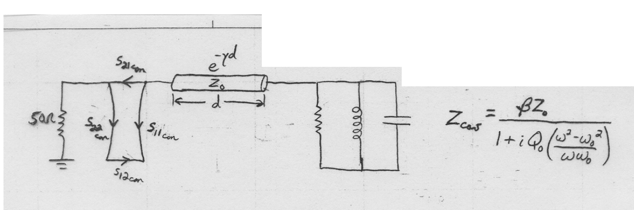

Figure 1: Measurement System Circuit Model

The unloaded Q of the cavity is typically measured as follows:

1.) *Find the resonant frequency,* ω_n *,of the system by maximizing the signal at the pickup probe and measure the electric field decay time,* $\tau_{E\,meas}$*, of the system while pulsing the cavity at* ω_n .

2.) *Measure the reflection coefficient,* Γ_{meas}*, on resonance at the measurement reference plane to determine the input coupling coefficient,* β_{meas}*, according to the formula*

$$\beta_{meas} = \frac{1 \pm |\Gamma_{meas}|}{1 \mp |\Gamma_{meas}|} \quad , \quad (1)$$

where the upper signs are chosen for the over-coupled case and the lower signs are chosen for the under-coupled case.

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[1] For the development of the circuit model for a cavity near resonance, see [1] Ch. 4 & 5and [2] Ch. 9.

[2] For a discussion of the properties of reciprocal 2-port networks see [3] pp.190-191,370-372 and [4] pp.199-201.

3.) Calculate the unloaded Q, $Q_{o\,meas}$, from the resonant frequency, the electric field decay time, and the input coupling coefficient according to the formula

$$Q_{o\,meas} = (1+\beta_{meas}) \cdot \omega_n \frac{\tau_{E\,meas}}{2} \quad . \qquad (2)$$

4.) Furthermore, the electric and magnetic field levels inside the cavity are calculated from the electric and magnetic energies, which are assumed to be equal at resonance and are calculated according to the formula

$$U_{E\,Peak} = U_{M\,Peak} = P_{cav\,meas} \cdot \frac{Q_{o\,meas}}{\omega_n} \qquad (3)$$

where $U_{E\,Peak}$ and $U_{M\,Peak}$ are the peak electric and magnetic energy respectively, and $P_{cav\,meas}$ is the measured power loss in the cavity which is determined from the power delivered to the network.

Solution Procedure

To quantify the effects of the reflective adapter on the system measurements, the following solution procedure was applied:

1.) Calculate the natural resonant frequency and the decay time of the un-driven system consisting of the parallel combination of the cavity impedance at the cavity reference plane and the impedance presented to the cavity by the input coupler.

The impedance presented to the cavity is the impedance seen looking from the cavity reference plane towards the source. Due to the reflective adapter, the 50Ω source impedance is transformed through the adapter and the transmission line into a reactive impedance at the cavity reference plane. Thus, at the cavity reference plane, the undriven system can be represented as in Fig. 2.

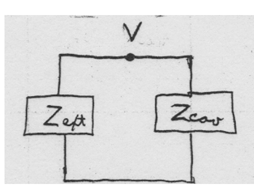


Figure 2:

In the absence of a source current, the following equation must be satisfied for the circuit of Fig. 2,

$$V\left(\frac{1}{Z_{ext}} + \frac{1}{Z_{cav}}\right) = 0 \ . \quad (4)$$

The solution for V is assumed to be of the form $Ve^{i\omega t}$. For a nontrivial solution to (4), we must have

$$Z_{ext} = -Z_{cav} \ , \quad (5)$$

which is satisfied for passive impedances by allowing ω to be complex. Thus, we define the complex frequency at which (5) is satisfied as

$$\omega_{decay} = \omega_n + i\frac{1}{\tau_E} \qquad (6)$$

where ω_n is the natural frequency of oscillations and τ_E is the decay time-constant of the voltage. The meaning becomes clear when considering that the solution for V becomes

$$V = Ve^{-\frac{t}{\tau_E}} \cdot e^{i \cdot \omega_n t} \ . \ (7)$$

Rewriting Eq. (5), ω_{decay} is found as the root of the following equation:

$$f(\omega,d,\beta) = \left[1 + iQ_o\frac{\omega^2 - \omega_o^2}{\omega \cdot \omega_o}\right] \cdot \left[\frac{1 + s_{11con}e^{-2\gamma(\omega)\cdot d}}{1 - s_{11con}e^{-2\gamma(\omega)\cdot d}}\right] + \beta \quad (8)$$

where d is the length of the transmission line with complex propagation constant $\gamma(\omega) = \alpha + i\frac{\omega}{c}$ with losses of α (Nepers/m), with an adapter which has a reflection coefficient of s_{11con} and onto which the cavity of resonant frequency ω_o and unloaded Q, Q_o, is coupled with a coupling coefficient β. The derivation of this equation and subsequent equations can be found in Appendix A.

It is at ω_n that the measurements are assumed to be taken because this is approximately equal to the frequency at which the maximum voltage and the maximum pickup probe signal will occur when the circuit of Fig. 2 is driven with a sinusoidal source. This approximation was investigated and was found to be valid.

Note: ω_n will be different from ω_o, the cavity's design resonant frequency, when the reflective adapter causes the impedance presented to the cavity to be reactive. Thus, the reflective adapter influences the frequency tuning of the cavity. This de-tuning is also dependent upon the transmission line length separating the cavity from the adapter and the coupling coefficient β.

The measured loaded quality factor of the system of Fig. 2 is thus calculated as,

$$Q_L(d,\beta) = \omega_n(d,\beta)\frac{\tau_E(d,\beta)}{2} \quad (9)$$

where as in Eq. (8), it has been allowed for d and β to be varied to simulate various physical situations.

2.) Calculate the reflection coefficient seen at the measurement reference plane at the natural resonant frequency and the corresponding coupling coefficient.

The reflection coefficient seen at the measurement reference plane can be expressed as

$$\Gamma_{meas}(\omega,d,\beta) = s_{11con} + \frac{(s_{21con})^2 \cdot \Gamma_{cav}(\omega,\beta)\cdot e^{-2\gamma(\omega)\cdot d}}{1 - s_{11con}\cdot \Gamma_{cav}(\omega,\beta)\cdot e^{-2\gamma(\omega)\cdot d}} \quad (10)$$

where s_{21con} is the transmission coefficient of the lossless reciprocal adapter given as

$$s_{21con} = \sqrt{1-(s_{11\,con})^2}\cdot e^{i\frac{\pi}{2}} \quad (11)$$

and where $\Gamma_{cav}(\omega,\beta)$ is the reflection coefficient of the cavity which is coupled to the transmission line with coupling coefficient, β, at the cavity reference plane and which is expressed as

$$\Gamma_{cav}(\omega,\beta) = \frac{\beta - \left[1 + i\,Q_o\,\dfrac{\omega^2-\omega_o^{\;2}}{\omega\omega_o}\right]}{\beta + \left[1 + i\,Q_o\,\dfrac{\omega^2-\omega_o^{\;2}}{\omega\omega_o}\right]}\,. \quad (12)$$

In (10) it is assumed the adapter is a lossless reciprocal 2-port and that the reference planes of the adapter are chosen such that s_{11con} is purely real for simplicity.

Once Γ_{meas} is known at $\omega_n(d,\beta)$, the measured coupling coefficient, β_{meas}, is calculated as

$$\beta_{meas}(d,\beta) = \frac{1 + sign\left(\dfrac{Q_o}{Q_L(d,\beta)} - 2\right)\cdot\left|\Gamma_{meas}(\omega_n,d,\beta)\right|}{1 - sign\left(\dfrac{Q_o}{Q_L(d,\beta)} - 2\right)\cdot\left|\Gamma_{meas}(\omega_n,d,\beta)\right|}\,, \quad (13)$$

where mathematically the choice of over-coupling or under-coupling is based upon whether the known unloaded Q, Q_o, is respectively either more than or less than twice the observed loaded Q, Q_L. In actual measurement practice this choice is based upon the signature of the time-domain reflected power seen at the measurement plane. This mathematical simplification eliminates having to know the adapter's s-parameters over a very broad range of frequency; which would be necessary to simulate a pulsed response.

3.) Calculate the measured unloaded Q and the associated percent error.

Given the results of the previous steps, the measured unloaded Q, $Q_{o\,meas}$, would be calculated as

$$Q_{o\,meas}(d,\beta) = \left[1 + \beta_{meas}(d,\beta)\right]\cdot Q_L(d,\beta)\,. \quad (14)$$

The resulting percent error between $Q_{o\,meas}$ and the actual Q_o is thus defined to be

$$\%Error_{Q_{o\,meas}} = \frac{Q_{o\,meas} - Q_o}{Q_o}\cdot 100 \quad (15)$$

4.) Furthermore, calculate the error in the measured cavity electric and magnetic energy.

The peak voltage appearing across the cavity for a unit peak forward voltage wave applied by the source at the measurement plane can be expressed as,

$$V_{cav}(\omega,d,\beta) = s_{21con}e^{-\gamma(\omega)\cdot d}\cdot\frac{1+\Gamma_{cav}(\omega,\beta)}{1 - s_{22con}e^{-2\gamma(\omega)\cdot d}\Gamma_{cav}(\omega,\beta)}\,. \quad (16)$$

The power delivered to the network at the natural resonant frequency, or the assumed power delivered to the cavity, P_{cav}, as measured at the measurement reference plane for a unit peak forward voltage wave is just the difference between the power associated with the forward wave and the power associated with the reflected wave at the measurement reference plane and is expressed as

$$P_{cav\ meas}(d,\beta) = \frac{1}{2Z_o} \cdot \left[1 - \left|\Gamma_{meas}(\omega_n(d,\beta),d,\beta)\right|^2\right] \quad (17)$$

where Z_o is the characteristic impedance of the transmission line separating the adapter and the cavity.

The assumed peak electric and magnetic energy will then be measured as,

$$U_{E\ meas} = U_{M\ meas} = P_{cav\ meas}(d,\beta) \cdot \frac{Q_{o\ meas}(d,\beta)}{\omega_n(d,\beta)} \quad . (18)$$

The actual peak electric and magnetic energy in the cavity are given as

$$U_E(d,\beta) = \frac{Q_o}{\omega_o} \cdot \frac{\left|V_{cav}(\omega_n(d,\beta),d,\beta)\right|^2}{2\beta Z_o} \quad , (19)$$

and

$$U_M(d,\beta) = \left(\frac{\omega_o}{\omega_n(d,\beta)}\right)^2 U_E(d,\beta) \quad . \quad (20)$$

Thus, the associated percent errors are,

$$\%Error\ U_{E\ meas} = \frac{U_{E\ meas} - U_E}{U_E} \cdot 100 \quad (21)$$

and

$$\%Error\ U_{M\ meas} \cong \%Error\ U_{E\ meas} \quad (22)$$

since $\left(\frac{\omega_o}{\omega_n(d,\beta)}\right)^2$ is approximately equal to 1.

Results

The solution procedure was performed for the parameters of the CKM vertical test stand Q-measurement system using Mathcad[3]. The parameters of the CKM cavities and the vertical test stand that were used in the equations are as follows:

$$Q_o = 2.1 \cdot 10^9$$
$$\omega_o = 2\pi f_o \text{ with } f_o = 3.9 \cdot 10^9$$
$$Q_{ext\ Design} = 6 \cdot 10^7 \Rightarrow \beta_{Design} = 35$$
$$\beta = 0 \ to\ 35$$

VSWR$_{adapter}$: 2.33 $\Rightarrow s_{11con} \cong 0.4 \cong -8\,dB$

α (transmission line attenuation): 0.03 Np/m

d_o (nominal transmission line length) $= 4\lambda_o$

$$d = d_o \ to\ (d_o + 0.5\lambda_o)$$

β was varied from 0 to 35, which corresponds to Q_{ext} ranging from infinite to the design Q_{ext} of $6 \cdot 10^7$ to simulate a range of various cavity coupling coefficients, including near critical coupling where typical Q-measurements are taken. Also the length d was varied over one half of a wavelength to simulate a range of transmission line lengths separating the cavity and the adapter.

The results of the solution with these parameters are shown in Figs. 3-6. A copy of the Mathcad input file used for the simulation is included in Appendix B.

[3] Mathcad is written by Mathsoft, Inc. 101 Main Street, Cambridge, MA 02142

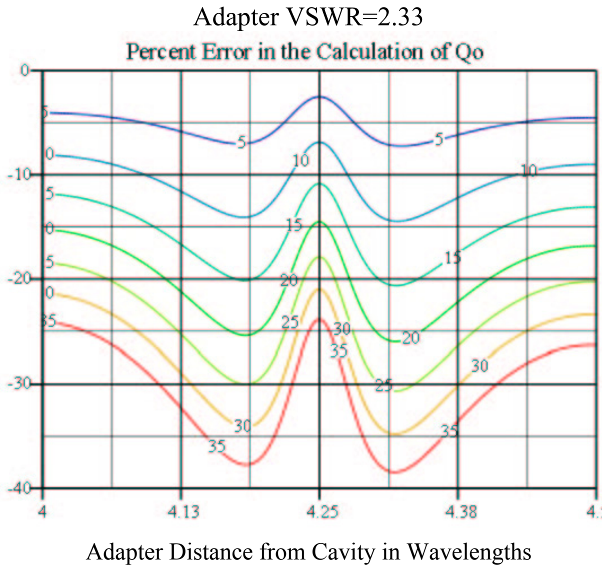


Figure 3: Calculated Percent Error in Measurement of Q_o for the CKM Vertical Test Stand with an adapter of VSWR=2.33 shown for constant cavity coupling coefficient contours, β (from 0 to 35), as a function of adapter distance from the cavity in cavity resonant frequeny wavelengths. Vertical axis is percent error.

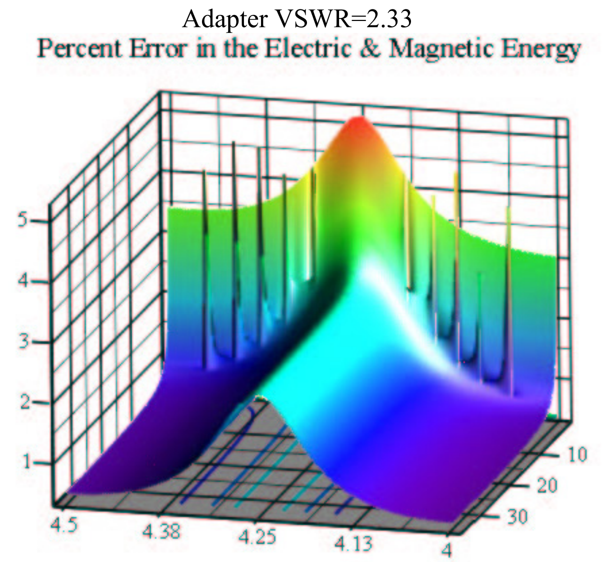


Figure 4: Percent Error (z-axis) in the Calculation of the Electric and Magnetic Energy for an adapter of VSWR=2.33 as a function of both cavity coupling coefficient, β (from 0 to 35 in the x-axis), and adapter distance from the cavity, d (from $4\lambda_o$ to $4.5\lambda_o$ in the y-axis). ***Note: The spikes occurring for low coupling coefficients are due to a mathematical switch in the decision of an over-coupled or an under-coupled situation.***

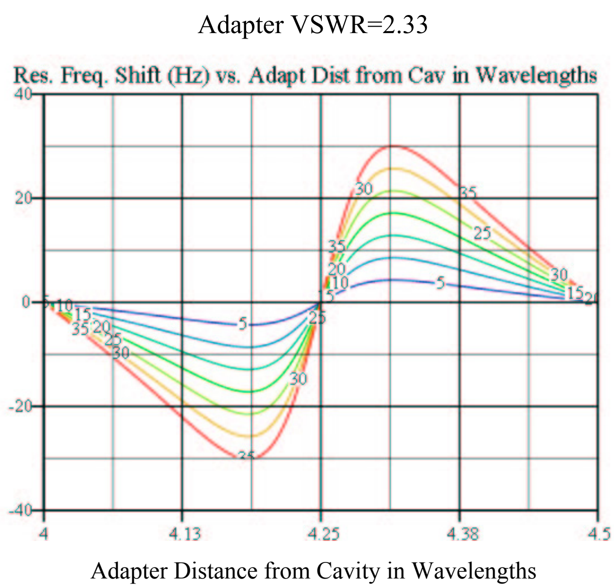


Figure 5: Resonant frequency shift in Hertz (y-axis) for an adapter of VSWR=2.33, shown for constant β contours as a function of the adapter distance from the cavity in wavelengths (x-axis). Vertical axis is resonant frequency shift in Hz.

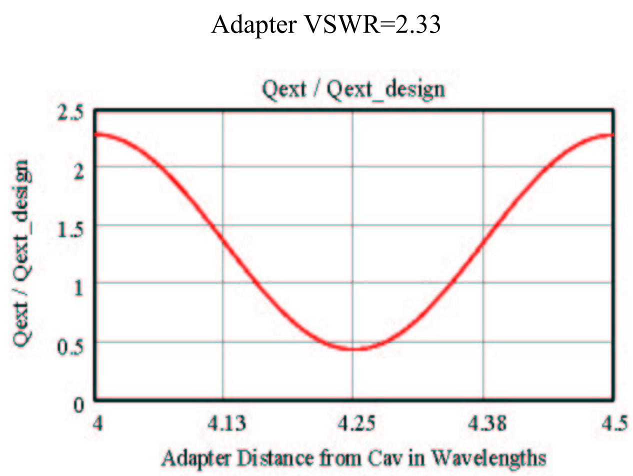


Figure 6: Ratio of the actual Qext to the designed Qext for an adapter of VSWR=2.33. The designed Qext represents the cavity coupling to the transmission line as designed by the antenna geometry, whereas the actual Qext represents the true loading of the cavity due to the reflective adapter. This curve was found to be true for all values of designed cavity coupling coefficient β.

As a further investigation, Fig. 7 shows the error which would result if the adapter was calibrated out for the steady-state measurement of the input coupling coefficient but not during the decay measurement. This is the situation that occurs in practice when the cavity is forced to see the reflective adapter during decay.

Percent Error with Steady-State Calibration But Without Decay Calibration

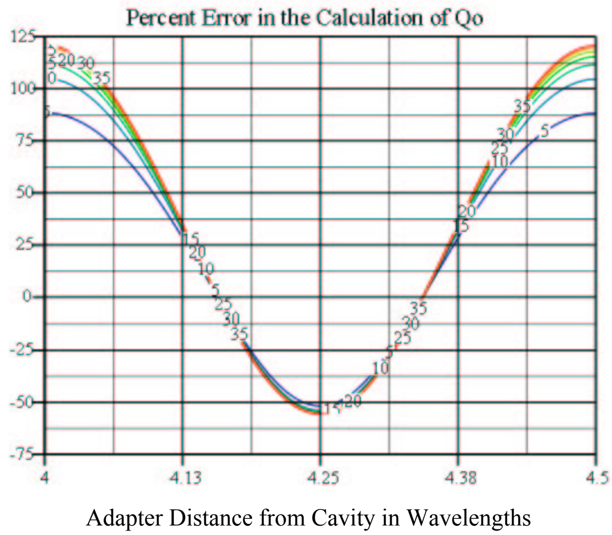


Figure 7: Resultant Percent Error in Q_o if the adapter is assumed to be calibrated out of the steady-state measurement of the coupling coefficient but cannot be calibrated out of the decay time for an adapter of VSWR=2.33.

Conclusion

The measurement error in the unloaded quality factor for the CKM vertical test stand Q-measurement system has been investigated. The error in the unloaded quality factor measurement is well under 10% for near unity coupling without performing a calibration of the reflective adapter. The error in the calculation of the cavity energy is under 5%. However, the reflective adapter does cause a resonant frequency shift of the system which is dependent upon the transmission line length which is varied to adjust the cavity coupling. These effects will grow with increasing VSWR of the adapter. Also not considered is the electromagnetic stress which the adapter will incur when placed in a transmission line with a high VSWR such as in the designed system which has a large cavity coupling coefficient by design.

The low percent error is a result of the short length of transmission line in the system. For longer lengths of transmission line separating the reflective interface from the cavity, the error will grow due to the fact that the reflective interface is closer to the measurement plane than to the cavity. Thus, the measurement of the input coupling coefficient is affected more greatly than the decay time measurement. In this situation, a calibration of the adapter may help the measurements. However, as was seen for the given conditions of the CKM Vertical Test Stand, a calibration of this adapter should not be performed, for it would cause larger error.

References

[1] J.C. Slater, “Microwave Electronics”, D. Van Nostrand Company Inc., 1950.

[2] E.L. Ginzton, “Microwave Measurements”, McGraw-Hill Book Co., 1957, Library of Congress Catalog Number 56-13393.

[3] R.E. Collins, “Field Theory of Guided Waves”, McGraw-Hill Inc., New York, 1960.

[4] D.M. Pozar, “Microwave Engineering”, 2nd Edition, J.Wiley&Sons, Inc., New York, 1998.

[5] G. Gonzalez, “Microwave Transistor Amplifiers”, Prentice-Hall, Engelwood Cliffs, 1984, Chapter 1.

Appendix A

The impedance of a parallel resonant circuit at the end of a transmission line can be written as:

$$Z_{cav} = \frac{\beta Z_o}{1 + i Q_o \left(\frac{\omega^2 - \omega_o^2}{\omega \omega_o} \right)} . \quad (1)$$

This presents a reflection coefficient at the end of a transmission line given as:

$$\Gamma_{cav}(\omega, \beta) = \frac{\beta - \left[1 + i Q_o \frac{\omega^2 - \omega_o^2}{\omega \omega_o} \right]}{\beta + \left[1 + i Q_o \frac{\omega^2 - \omega_o^2}{\omega \omega_o} \right]} \quad (2)$$

It is from this equation that the coupling coefficient can be calculated from a measurement of the magnitude of the reflection coefficient at ω_o for which

$$\beta_{meas} = \frac{1 \pm |\Gamma_{meas}|}{1 \mp |\Gamma_{meas}|} \quad (3)$$

where the upper signs are chosen for the overcoupled case ($\beta > 1$) and the lower signs are chosen for the undercoupled case ($\beta < 1$).

The reflection coefficient of Eq.(2) is transformed through a lossy transmission line and presents a reflection coefficient to the output of a connector given as:

$$\Gamma_L(\omega, \beta) = \Gamma_{cav}(\omega, \beta) \cdot e^{-2\gamma(\omega) \cdot d} \quad (4)$$

Using s-parameter theory, the input reflection seen at the input of a connector as shown in Fig. A1 can be expressed as:

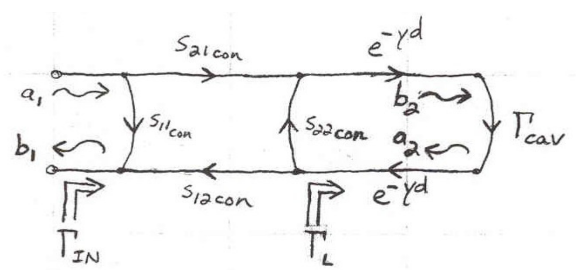


Figure A1

$$\Gamma_{IN} = s_{11con} + \frac{s_{12con} s_{21con} \Gamma_L}{(1 - s_{22con} \Gamma_L)} \quad (5)$$

If the connector is assumed to be lossless, reciprocal and physically symmetric, one can take advantage of the unitary properties of the connector's s-matrix (see text footnote 2). Under such conditions:

$$s_{12con} = s_{21con} \quad (6a)$$
$$s_{11con} = s_{22con} \quad (6b)$$

Furthermore, if it is assumed that the connector reference plane is such that s11 is real for simplicity, then the amplitude and phase constraint on s21 for the connector's s-matrix to be unitary is such that

$$s_{21con} = \sqrt{1 - (s_{11con})^2} \cdot e^{i\frac{\pi}{2}} \quad (7)$$

Equations (4)-(6) can be combined to yield the expression for the reflection coefficient seen at the measurement plane

$$\Gamma_{meas}(\omega, d, \beta) = s_{11con} + \frac{(s_{21con})^2 \cdot \Gamma_{cav}(\omega, \beta) \cdot e^{-2\gamma(\omega) \cdot d}}{1 - s_{11con} \cdot \Gamma_{cav}(\omega, \beta) \cdot e^{-2\gamma(\omega) \cdot d}} \quad (8)$$

The definition of reflection coefficient in terms of impedances is given as (see Ref [5])

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} \quad (9)$$

where Z_L is the load impedance at the end of a transmission line of characteristic impedance Z_o. From this equation the impedance presented to the parallel resonant circuit of Fig. 1 from the text can be calculated. The matched source impedance is transformed through the connector's s-matrix and the lossy transmission line such that the reflection coefficient seen on the lossy transmission line looking out from the parallel resonant circuit is given as

$$\Gamma_{ext} = s_{11con} \, e^{-2\gamma(\omega) \cdot d} \quad (10)$$

where use of Eq. (5) was made with $\Gamma_L = 0$ due to the source match. Using equation (9), the impedance associated with this reflection coefficient is expressed as

$$Z_{ext} = \frac{1+s_{11con}\, e^{-2\gamma(\omega)\cdot d}}{1-s_{11con}\, e^{-2\gamma(\omega)\cdot d}} \cdot Z_o \quad (11)$$

This impedance appears in parallel with the parallel resonant circuit such as in Fig. 2 of the text. Thus, in the absence of a source impedance, Kirchhoff's current law stipulates that

$$V\left(\frac{1}{Z_{ext}} + \frac{1}{Z_{cav}}\right) = 0 \quad (12)$$

As in the text, the solution for the circuit voltage, V, is assumed to be of the form $Ve^{i\omega t}$ for which a complex frequency allows equation (12) to be satisfied by making

$$Z_{ext} = -Z_{cav} \quad (13)$$

Using Eq. (1) for the cavity impedance, Z_{cav}, and Eq. (11) for the external impedance, Z_{ext}, Eq. (13) can be written as

$$\left[1+iQ_o\frac{\omega^2-\omega_o{}^2}{\omega\cdot\omega_o}\right]\cdot\left[\frac{1+s_{11con}e^{-2\gamma(\omega)\cdot d}}{1-s_{11con}e^{-2\gamma(\omega)\cdot d}}\right]+\beta=0 \ . \ (14)$$

And thus, the complex resonant frequency which gives the natural frequency of oscillations and the decay time-constant is found to be the root of the function,

$$f(\omega,d,\beta)=\left[1+iQ_o\frac{\omega^2-\omega_o{}^2}{\omega\cdot\omega_o}\right]\cdot\left[\frac{1+s_{11con}e^{-2\gamma(\omega)\cdot d}}{1-s_{11con}e^{-2\gamma(\omega)\cdot d}}\right]+\beta \ . \ (15)$$

The solution of the voltage at the cavity reference plane when the system is driven can be carried out with reference to Fig. A1. Using voltage waves, the voltage wave traveling towards the cavity in Fig. A1 is expressed in terms of the other waves as

$$b_2 = s_{21con}e^{-\gamma(\omega)\cdot d}a_1 + s_{22con}e^{-2\gamma(\omega)\cdot d}a_2 \ . \ (16)$$

The wave reflected off of the cavity is expressed as

$$a_2 = \Gamma_{cav}\, b_2 \ . \ (17)$$

Combining (16) and (17) yields

$$b_2 = \frac{s_{21con}e^{-\gamma(\omega)\cdot d}}{(1-s_{22con}e^{-2\gamma(\omega)\cdot d}\Gamma_{cav})}a_1 \ . \ (18)$$

The total voltage at the cavity reference plane is the sum of the forward and reflected waves

$$V_{cav} = a_2 + b_2 \ . \quad (19)$$

Substituting Eq.'s (17) and (18) into Eq. (19) yields

$$V_{cav} = a_1\cdot s_{21con}e^{-\gamma(\omega)\cdot d}\ \frac{(1+\Gamma_{cav})}{(1-s_{22con}e^{-2\gamma(\omega)\cdot d}\Gamma_{cav})} \ . \ (20)$$

On resonance, the electric and magnetic energy of a parallel RLC circuit are equal.

$$U_E = \frac{1}{2}CV_{pk}^2 \quad (21)$$

$$U_M = \frac{1}{2}LI_{pk}^2 \quad (22)$$

$$U_E = U_M \quad (23)$$

where

$$I_{pk} = \left|\frac{V_{pk}}{i\omega L}\right| \quad (24)$$

is the peak current in the inductor and V_{pk} is the voltage appearing across the RLC circuit. Since

$$L = \frac{1}{\omega_o{}^2C} \ , \ (25)$$

Eq. (22) can be written using Eq. (24) & (25) as

$$U_M = \left(\frac{\omega_o}{\omega}\right)^2 U_E \quad (26)$$

while Eq. (21) can be rewritten using $Q_o = \omega_o RC$ as

$$U_E = \frac{Q_o}{2\omega_o R}\cdot V_{pk}^2 \ . \ (27)$$

Appendix B: Mathcad Input File

Effect of a Reflective Interface on the Q-measurements of a Resonant Circuit

First define our systems parameters

Define our adapter's s-parameters

$VSWR := 2.33 \qquad s11_{con} := \frac{VSWR - 1}{VSWR + 1} \qquad s22_{con} := s11_{con} \qquad s21_{con} := \sqrt{1 - \left(\left|s11_{con}\right|\right)^2} \cdot e^{i \cdot \frac{\pi}{2}}$

$s11_{con} = 0.399 \quad s22_{con} = 0.399 \quad s21_{con} = 0.917i \qquad 20 \cdot \log\left(s11_{con}\right) = -7.972$

Define the cavity parameters:

$f_o := 3.9 \cdot 10^9 \qquad \omega_o := 2\pi \cdot f_o \qquad Q_o := 2.1 \cdot 10^9$

Next define the transmission line properties. The Vertical Test Stand exhibits approximately 4 wavelengths between the cavity and the adapter.

$c := 3 \cdot 10^8 \qquad \lambda_o := \frac{c}{f_o} \qquad Z_o := 50$

$l_{ant} := 4 \cdot \lambda_o \qquad \alpha_{ant} := 0.03 \qquad \gamma_{ant}(\omega) := \alpha_{ant} + i \cdot \frac{\omega}{c}$

Now set up arrays for varying the length and the cavity coupling coefficient

$num_{len} := 160 \qquad num_{\beta} := 100 \qquad n := 0, 1 .. num_{len} \qquad m := 0, 1 .. num_{\beta}$

$Q_{ext\_min} := 6 \cdot 10^7 \qquad \beta_{max} := \frac{Q_o}{Q_{ext\_min}} \qquad len_n := l_{ant} + \frac{n}{num_{len}} \cdot \frac{\lambda_o}{2} \qquad \beta_{in_m} := 0 + \frac{m}{num_{\beta}} \cdot \beta_{max}$

Define a frequency vector for a frequency sweep:

$freq\_start := 3.8999999 \cdot 10^9 \qquad freq\_stop := 3.9000001 \cdot 10^9 \qquad num\_pts := 801$

$k := 0, 1 .. num\_pts - 1$

$freq\_step := \frac{freq\_stop - freq\_start}{num\_pts - 1}$

$freq_k := freq\_start + k \cdot freq\_step \qquad \Delta freq_k := freq_k - f_o$

Define all functions:

1.) Calculate the natural resonant frequency and the decay time of the un-driven system consisting of the parallel combination of the cavity impedance at the cavity reference plane and the impedance pressented to the cavity by the input coupler.

Define the function for which we are to solve for complex ω.

$$f(\omega, d, \beta) := \left[1 + i \cdot Q_o \cdot \left(\frac{\omega^2 - \omega_o^2}{\omega \cdot \omega_o}\right)\right] \cdot \frac{\left(1 + s11_{con} \cdot e^{-2\gamma_{ant}(\omega) \cdot d}\right)}{\left(1 - s11_{con} \cdot e^{-2\gamma_{ant}(\omega) \cdot d}\right)} + \beta$$

The complex resonant frequency which solves the circuit equations is the root of this function. To find the root, we need an initial guess.

$$\omega := \omega_o + i$$

$$\omega_{decay}(d, \beta) := \mathrm{root}(f(\omega, d, \beta), \omega) \quad \omega_n(d, \beta) := \mathrm{Re}\left(\omega_{decay}(d, \beta)\right) \quad \tau_E(d, \beta) := \frac{1}{\mathrm{Im}\left(\omega_{decay}(d, \beta)\right)}$$

Now calculate the loaded Q which is measured during the cavity decay

$$Q_L(d, \beta) := \omega_n(d, \beta) \cdot \frac{\tau_E(d, \beta)}{2} \qquad \beta_{decay}(d, \beta) := \frac{Q_o}{Q_L(d, \beta)} - 1$$

2.) Calculate the reflection coefficient seen at the measurement reference plane at the natural resonant frequency and the corresponding coupling coefficient

First find the reflection coefficient at the cavity plane:

$$\Gamma_{cav}(\omega, \beta) := \frac{\beta - \left(1 + i \cdot Q_o \cdot \frac{\omega^2 - \omega_o^2}{\omega \cdot \omega_o}\right)}{\beta + \left(1 + i \cdot Q_o \cdot \frac{\omega^2 - \omega_o^2}{\omega \cdot \omega_o}\right)}$$

Then calculate the reflection coefficient at the measurement plane:

$$\Gamma_{meas}(\omega, d, \beta) := s11_{con} + \frac{s21_{con}^2 \cdot \Gamma_{cav}(\omega, \beta) \cdot \left(e^{-2\gamma_{ant}(\omega) \cdot d}\right)}{1 - s11_{con} \cdot \Gamma_{cav}(\omega, \beta) \cdot \left(e^{-2\gamma_{ant}(\omega) \cdot d}\right)}$$

We base our mathematical decision of over-coupling or under-coupling upon the sign of the coupling that the cavity sees during decay.

$$\beta_{meas}(d, \beta) := \frac{1 + \mathrm{sign}\left(\frac{Q_o}{Q_L(d, \beta)} - 2\right) \left|\Gamma_{meas}\left(\omega_n(d, \beta), d, \beta\right)\right|}{1 - \mathrm{sign}\left(\frac{Q_o}{Q_L(d, \beta)} - 2\right) \left|\Gamma_{meas}\left(\omega_n(d, \beta), d, \beta\right)\right|}$$

3.) Calculate the measured unloaded Q and the associated percent error.

$$Q_{o\_meas}(d,\beta) := \left(1+\beta_{meas}(d,\beta)\right)\cdot Q_L(d,\beta) \qquad Q_{\%error}(d,\beta) := \frac{Q_{o\_meas}(d,\beta)-Q_o}{Q_o}\cdot 100$$

4.) Furthermore, calculate the error in the measured cavity electric and magnetic energy.

Calculate the voltage appearing across the cavity impedance:

$$V_{cav}(\omega,d,\beta) := \left| s21_{con}\cdot\left(e^{-\gamma_{ant}(\omega)\cdot d}\right)\cdot\frac{\left(1+\Gamma_{cav}(\omega,\beta)\right)}{\left[1-s22_{con}\cdot\left(e^{-2\gamma_{ant}(\omega)\cdot d}\right)\cdot\Gamma_{cav}(\omega,\beta)\right]}\right|$$

Calculate the assumed power delivered to the cavity:

$$P_{cav\_meas}(d,\beta) := \frac{1}{2\cdot Z_o}\cdot\left[1-\left(\left|\Gamma_{meas}\left(\omega_n(d,\beta),d,\beta\right)\right|\right)^2\right]$$

Calculate the assumed electric and magnetic energy:

$$U_{cav\_meas}(d,\beta) := P_{cav\_meas}(d,\beta)\cdot\frac{Q_{o\_meas}(d,\beta)}{\omega_n(d,\beta)}$$

Calculate the actual electric and magnetic energy:

$$U_E(d,\beta) := \frac{Q_o}{\omega_o}\cdot\frac{\left(\left|V_{cav}\left(\omega_n(d,\beta),d,\beta\right)\right|\right)^2}{2\cdot\beta\cdot Z_o} \qquad U_M(d,\beta) := \frac{\omega_o}{\left(\omega_n(d,\beta)\right)^2}\cdot U_E(d,\beta)$$

Calculate the percent error in the electric and magnetic energy, assume ω_n ~= ω_o :

$$U_{E\_\%error}(d,\beta) := \frac{U_{cav\_meas}(d,\beta)-U_E(d,\beta)}{U_E(d,\beta)}\cdot 100 \qquad U_{M\_\%error}(d,\beta) := U_{E\_\%error}$$

Now that all functions have been defined, let's calculate them for our systems parameters:

Note: For some calculations we need not calculate from the function but from previous calculations. This saves calculation time.

$$\omega_{complex_{n,m}} := \omega_{decay}\left(len_n, \beta in_m\right) \qquad f_{shift_{n,m}} := \frac{Re\left(\omega_{complex_{n,m}}\right) - \omega_o}{2\pi}$$

$$\beta_{decay\_calc_{n,m}} := \beta_{decay}\left(len_n, \beta in_m\right)$$

$$Q_{o\_calc_{n,m}} := Q_{o\_meas}\left(len_n, \beta in_m\right) \qquad Q_{\%error\_calc_{n,m}} := \frac{Q_{o\_calc_{n,m}} - Q_o}{Q_o} \cdot 100$$

$$Q_{ext_{n,m}} := \frac{Q_o}{\beta_{decay\_calc_{n,m}}} \qquad Q_{ext\_factor_{n,m}} := \frac{\beta in_m}{\beta_{decay\_calc_{n,m}}}$$

$$V_{cav\_calc_{n,m}} := V_{cav}\left(Re\left(\omega_{complex_{n,m}}\right), len_n, \beta in_m\right)$$

*Calculate the normalized power that is delivered to the entire network. P forward is normalized to (Vfwd=1)^2 / (2) as opposed to Vfwd^2/(2*Zo) since Zo will drop out of the error expression.*

$$\Gamma_{meas\_fo\_calc_{n,m}} := \Gamma_{meas}\left(Re\left(\omega_{complex_{n,m}}\right), len_n, \beta in_m\right)$$

$$P_{cav\_calc_{n,m}} := \frac{1}{2}\cdot\left[1 - \left(\left|\Gamma_{meas\_fo\_calc_{n,m}}\right|\right)^2\right]$$

Now, based upon this power, we calculate the normalized stored energy in the cavity from our calculations of Q and our driving frequency. Note: normalization to Zo, since P is normalized.

$$U_{cav\_calc_{n,m}} := \frac{P_{cav\_calc_{n,m}} \cdot Q_{o\_calc_{n,m}}}{Re\left(\omega_{complex_{n,m}}\right)}$$

$$U_{E\_calc_{n,m}} := \frac{Q_o}{2\cdot\omega_o}\cdot\frac{\left(V_{cav\_calc_{n,m}}\right)^2}{\beta in_m} \qquad U_{M\_calc_{n,m}} := \left(\frac{\omega_o}{Re\left(\omega_{complex_{n,m}}\right)}\right)^2 \cdot U_{E\_calc_{n,m}}$$

Cannot calculate at βin=0 since here U=0, thus let's only use non-zero values of β

$$m2 := 0, 1 .. num_\beta - 1$$

$$U_{E\_\%error\_calc_{n,m2}} := \frac{U_{cav\_calc_{n,m2+1}} - U_{E\_calc_{n,m2+1}}}{U_{E\_calc_{n,m2+1}}} \cdot 100$$

$$U_{M\_\%error\_calc_{n,m2}} := \frac{U_{cav\_calc_{n,m2+1}} - U_{M\_calc_{n,m2+1}}}{U_{M\_calc_{n,m2+1}}} \cdot 100$$

Error Check: Is the solution for $\omega_{complex}$ truly a root of the original equation?

$$zero_{n,m} := f\left(\omega_{complex_{n,m}}, len_n, \beta_{in_m}\right)$$

$$zero_{real\_abs_{n,m}} := \left|Re\left(zero_{n,m}\right)\right| \qquad zero_{imag\_abs_{n,m}} := \left|Im\left(zero_{n,m}\right)\right|$$

$$max\left(zero_{real\_abs}\right) = 1.381 \times 10^{-6} \qquad max\left(zero_{imag\_abs}\right) = 1.958 \times 10^{-6}$$

Define some parameters for facilitating plotting:

$$len2_{n,m} := \frac{len_n}{\lambda_o} \qquad \beta2_{n,m} := \beta_{in_m} \qquad \Delta freq2_{k,n} := \Delta freq_k$$

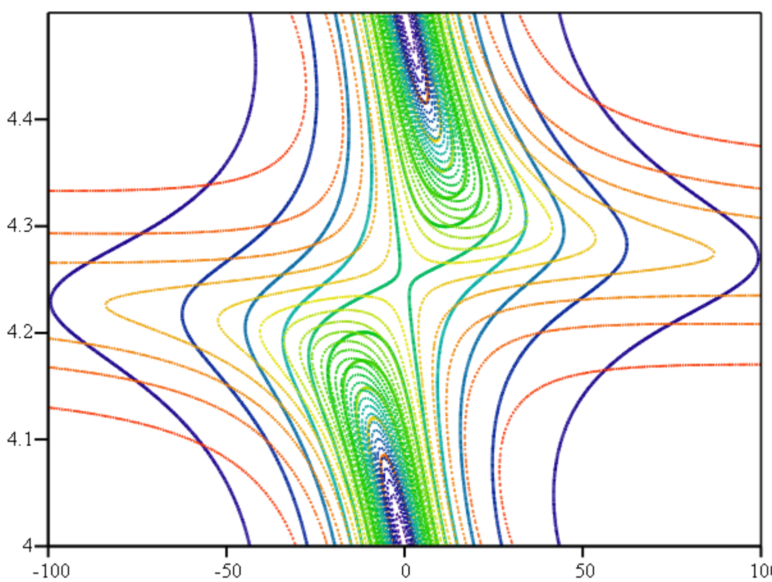
$$len3_{k,n} := \frac{len_n}{\lambda_o} \qquad len4_{n,m2} := \frac{len_n}{\lambda_o} \qquad \beta4_{n,m2} := \beta_{in_{m2+1}}$$

Let's perform some further investigations into overall circuit behavior:

Look at the response at a particular cavity coupling factor for all frequencies near resonance and for all adapter distances varying over 1/2 wavelength.

$$index_\beta := 40 \qquad V_{cav\_calc2_{k,n}} := V_{cav}\left[2\pi \cdot freq_k, len_n, \beta_{in_{\left(index_\beta\right)}}\right]$$

$$\Gamma_{cav\_calc_{k,m}} := \Gamma_{cav}\left(2\pi \cdot freq_k, \beta_{in_m}\right) \qquad \Gamma_{meas\_calc_{k,n}} := \left|\Gamma_{meas}\left[2\pi \cdot freq_k, len_n, \beta_{in_{\left(index_\beta\right)}}\right]\right|$$

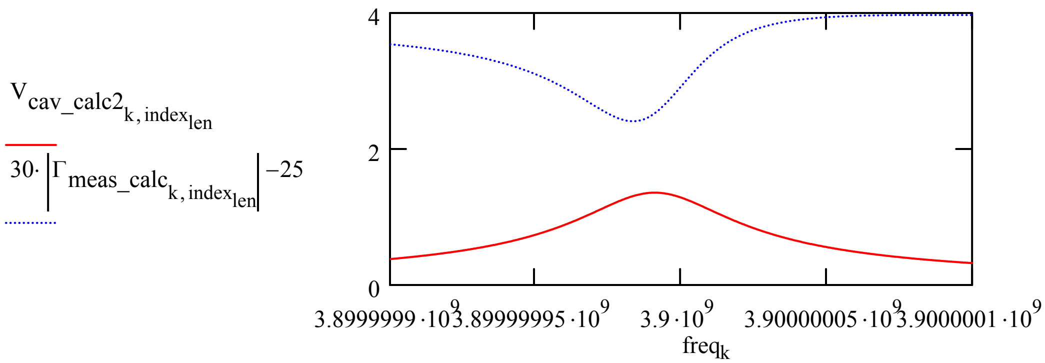


$$\left(\Delta freq2, len3, V_{cav\_calc2}\right), \left(\Delta freq2, len3, \Gamma_{meas\_calc}\right)$$

This plot shows that minimum reflection and maximum transmission do not always occur at the same frequency. This is seen by noticing that the contours of the voltage and the reflection coefficient are not overlapping for all conditions. The next section will look at a specific instance.

Let's compare frequencies where maximum cavity voltage and minimum reflection occur for a specific Adapter distance from the cavity.

$index_{len} := 70$



Find frequency of maximum cavity voltage:

$V_{cav\_test_k} := V_{cav\_calc2}(k, index_{len})$

$V_{max} := \max(V_{cav\_test})$ $\quad V_{max} = 1.35675$

$Index_{Vmax} := \mathrm{match}(V_{max}, V_{cav\_test})$

$f_{Vmax} := freq_{\left(Index_{Vmax_{round\left(\frac{rows(Index_{Vmax})}{2}\right)}}\right)}$

$f_{Vmax} = 3.899999991 \times 10^9$

Find frequency of minimum reflection:

$\Gamma_{meas\_test_k} := \left|\Gamma_{meas\_calc}(k, index_{len})\right|$

$\Gamma_{min} := \min(\Gamma_{meas\_test})$ $\quad \Gamma_{min} = 0.914$

$Index_{\Gamma min} := \mathrm{match}(\Gamma_{min}, \Gamma_{meas\_test})$

$f_{\Gamma min} := freq_{Index_{\Gamma min_{round\left(\frac{rows(Index_{\Gamma min})}{2}\right)}}}$

$f_{\Gamma min} = 3.899999983 \times 10^9$

Difference between frequency of maximum cavity voltage and minimum reflection:

$$f_{Vmax} - f_{\Gamma min} = 8$$

Compare frequency of maximum cavity voltage and the natural resonant frequency

$$f_{natural} := \frac{\mathrm{Re}\left[\omega_{decay}\left[len(index_{len}), \beta_{in}(index_{\beta})\right]\right]}{2\pi}$$

$$f_{natural} - f_{Vmax} = -0.1 \quad \text{Hz}$$